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Quantum mechanics on Riemannian manifold in Schwinger's quantization approach I

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Abstract. Schwinger's quantization scheme is extended in order to solve the problem of the formulation of quantum mechanics on a space with a group structure. The importance of Killing vectors in the quantization scheme is shown. Usage of these vectors makes the algebraic properties of the operators consistent with the geometrical structure of the manifold. The procedure of the definition of the quantum Lagrangian of a free particle and the norm of the velocity (momentum) operators is given. These constructions are invariant under a general coordinate transformation. The unified procedure for constructing the quantum theory on a space with a group structure is developed. Using this, quantum mechanics on a Riemannian manifold with a simply transitive group acting on it is investigated.

1 Introduction

There are many works [1–13] in which the authors have presented the formulations of quantum mechanics on spaces with a group structure. These investigations have been carried out not only for the sake of their academic interest, but also for particular applications. The development of quantum mechanics on curved spaces has been essentially stimulated by fruitful investigations of several non-linear models such as the quantized Skyrme model of the simplest baryons [6–9], the theory of three dimensional quantum chiral solitons [10, 11] and supersymmetrical [12] Fermi solitons. The main interest in these works lies in the problem of the determination of a so-called "quantum potential" and the gauge structure of quantum mechanics on a curved space (or a surface embedded into the Euclidean space). The operator ordering problem in the starting Lagrangian has not been exhaustively analyzed in these works. Moreover, some assumptions (which seem to be adequate) are used in explicit or hidden form without any rigorous definition.

In this connection we try to examine the complex of problems associated with quantum mechanics on a curved space equipped with a Riemannian metric and to give (as far as possible) the rigorous motivation of it. Our version of the formulation of quantum mechanics on a curved space is based on Schwinger's action principle [14]. Developing a quantization procedure, we show that permissible variations (appearing in the action principle) on such a manifold coincide with the Killing vectors which represent the group of isometries for the Riemannian metric. Hence, the quantum-mechanical properties of the theory (operator algebra and gauge structure) are determined by this group.

To develop the quantization procedure one at first has to choose the Lagrangian operator. When the quantum theory is constructed from the classical one, the canonical quantization is performed by the replacement of the functions by the corresponding operators in the algebraic expressions and writing down the commutation relations between the canonically conjugated variables as the replacement of the Poisson (or Dirac) brackets. In this case there appears an operator ordering problem because of the presence of non-commutative multipliers in the products, so that there are several operator functions that tend to the same classical expressions when $\hbar \to 0$. Therefore, the quantum theory cannot be constructed uniquely from the corresponding classical one if the usage of the quantization procedure is limited by the formal replacement.

In order to fix the operator ordering ambiguity one can find the restriction that motivates the choice of the quantum Lagrangian. At this stage the special features of the theory under consideration should be used.

In the present series of papers we perform the action principle quantization of the model with the classical Lagrangian $L_0 = (1/2)\dot{q}^\mu g_{\mu\nu}(q)\dot{q}^\nu$, which describes a freely

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moving particle on the Riemannian manifold M equipped by the metric $g_{\mu\nu}(q)$ (here q^μ are the local coordinates on M). While the operators q^μ are independent from each other, they can be viewed as the complete set of commuting observables (see, for example, [15]). Therefore, the theory must be invariant under a general coordinate transformation $q^\mu \to \overline{q}^\mu = \overline{q}^\mu(q)$, i.e. for the quantum Lagrangian we can write $L(q) = \overline{L}(\overline{q})$. Making the requirement that $L(q)$ tends to $L_0(q)$ when $\hbar \to 0$, we conclude that $L(q)$ has the same structure as $L_0(q)$ to order $\hbar^0$, but these functions differ in the term that includes $\hbar$ as an external multiplier. The commutation relations can be calculated by means of the action principle and, using them, we can find the total expression of the Lagrangian, constructed as a generalized scalar product of the velocity operator by itself, $L(q) = (1/2)(\dot{q}, \dot{q})$. This definition fixes the form of the quantum Lagrangian; its choice is made in terms of the equivalence principle and the physical meaning of the complete set of commuting observables. The use of this procedure enables one to treat the "quantum potential" as a correction in the quantum Lagrangian which makes it a quantum scalar under a general coordinate transformation.

Our results, obtained by means of this procedure, are represented in three papers. In the present work (the first of these) we introduce the main principles and the extension of Schwinger's quantization procedure for the case of a Riemannian space which is applied in order to construct quantum mechanics on a manifold with the simply transitive group of isometries acting on it. Here Killing vectors form the representation of the Lie algebra and its number is equal to the dimension of a manifold. In this case Schwinger's quantization procedure is realized without any difficulties and the results obtained are in accordance with [1], where the canonical quantization approach is used.

In a second paper we will consider the more complicated case of quantum mechanics on a homogeneous Riemannian manifold V_n. The number of Killing vectors is higher than the dimension of the manifold and they are not independent in a point $q_0 \in V_n$. Some of them form the representation of the isotropy group of $q_0 \in V_n$. It turns out that quantum mechanics on the homogeneous Riemannian manifold has a gauge structure and the isotropy group acts as the group of local gauge transformations. A gauge-fixing condition (which is necessary in this case) requires the configuration space to be extended by adding new coordinates. The quantum Lagrangian has to be modified by introducing new terms (depending on new degrees of freedom) in such a way that local gauge transformations become global ones. The results obtained are in accordance with [13].

In the third work we will consider the formulation of quantum mechanics on a Riemannian manifold with an intransitive group of isometries. In this case the number of Killing vectors is lower than the dimension of the manifold. It will be shown that the quantum dynamics is completely determined only for degrees of freedom which describe the invariant subspace (associated with Killing vectors). The other equations contain a gauge structure and a scalar "quantum potential" which indicates some arbitrariness in the theory. The formal results of this part need to be reexamined in order to establish the physical meaning and to find concrete applications.

In the final (fourth) paper we will expand the extended Schwinger quantization scheme to the case of superspace (considered as a quotient space $SP_4/SO(1,3)$).

The investigations performed in the present series of works shows that Schwinger's quantization scheme, extended for the case of manifolds with a group structure (including a superspace), may be viewed as a method of constructing quantum theory that generalizes the canonical quantization procedure. This method allows one to analyze the quantum-mechanical model in cases where the theory essentially depends on the geometrical structure of the manifold. As the canonical quantization procedure, it requires external motivation of the choice of the form of the quantum Lagrangian.

In our opinion the approach introduced in the present works can be useful for analysis models describing the particle-like solitons in the collective coordinate formalism [2–5].

2 Variation principle in quantum mechanics

The variational principle, adapted to the purposes of quantum mechanics, was investigated by Schwinger in 1951 [14]. In these works Schwinger analyzed the special case of a theory characterized by a Lagrangian with a linear kinetic part depending on generalized velocities $\{\dot{q}^\mu : \mu = \overline{1,n}\}$ (here $\{q^\mu : \mu = \overline{1,n}\}$ are the generalized coordinates of the dynamical system). He showed that the Heisenberg equations of motion and commutation relations consistent with them can be obtained within the framework of a unique scheme.

Schwinger's quantization approach is based on the assumption of the existence of a hermitean operator of the action functional $S[q, \dot{q}]$. Using this functional the variation of the propagator, caused by the infinitesimal unitary transformation of the complete set of commuting observables $\{\alpha\}$ can be defined by

$$\delta \langle \alpha_1, t_1 | \, \alpha_2, t_2 \rangle = \frac{\mathrm{i}}{\hbar} \langle \alpha_1, t_1 | \, \delta S[q, \dot{q}] \, | \alpha_2, t_2 \rangle , \qquad (2.1)$$

where $|\alpha_{1,2}, t_{1,2}\rangle$ are the initial and final states of the dynamical system (i.e. eigenvectors of operators of the complete set $\{\alpha\}$ for the moments of time $t_{1,2}$). The variation of the action functional $S[q, \dot{q}]$ satisfies the relation

$$\delta S[q, \dot{q}] = G(t_2) - G(t_1), \qquad (2.2)$$

where $G = G(t)$ is the hermitean generator of a unitary transformation.

As far as the variation of the action functional is completely determined by the variations of generalized coordinates $\{q^\mu\}$ and time t, the condition (2.2) establishes the connection between the infinitesimal unitary transformation in (2.1) and the variations δq and δt. Variations that satisfy (2.2) are called *permissible variations*[1].

Taking into account (2.1) and (2.2) and the unitary nature of the permissible variations one can draw the conclusion that the variation δA of an arbitrary operator A caused by the variations $\delta q, \delta t$ is determined by the following expression:

$$\delta A = \frac{1}{\mathrm{i}\hbar}\left[A, G\right]. \tag{2.3}$$

As far as in (2.3) δA and G contain variations $\delta q, \delta t$, this relation can be interpreted in two ways. If the commutation relations of the model are known, one can find the explicit form of the variations δq, δt from (2.3). On the other hand, if δq and δt are given, the relation (2.3) can be viewed as the condition that determines the algebra of the commutation relations of the model. In Schwinger's quantization scheme one uses exactly the second variant of the interpretation of the relation (2.3).

Further we extend Schwinger's quantization scheme to the case of a non-linear stationary model, in which the kinetic part of the action depends not only on the velocities $\{\dot{q}^\mu : \mu = \overline{1,n}\}$, but also on the coordinates $\{q^\mu : \mu = \overline{1,n}\}$. This fact necessarily causes an operator ordering problem in the Lagrangian which contains non-commutative operators q and $\dot{q}$. The Lagrangian determines the action by

$$S[q, \dot{q}] = \int_{t_1}^{t_2} L(q, \dot{q})\mathrm{d}t. \tag{2.4}$$

The action functional (2.4) is the starting concept in Schwinger's scheme. To investigate the physical meaning of the theory based on the action (2.4) with a given Lagrangian, one should make some a priori assumptions about the properties of the operator variables q and $\dot{q}$ (see Sect. 4 for further discussion).

To determine the form of the generator $G = G(t)$ of the unitary transformation, let us consider the special case of the coordinate transformation

$$\overline{q}(t) = q(t) + \delta q(t). \tag{2.5}$$

According to the properties of the canonical transformations, the Lagrangian can be expressed in terms of kinetic and dynamic parts:

$$L = L_{\mathrm{kin}} - H. \tag{2.6}$$

The variation of the kinetic part, L_{kin}, under the transformation (2.5) satisfies the condition

$$\delta L_{\mathrm{kin}} = -\frac{\mathrm{d}K}{\mathrm{d}t}, \tag{2.7}$$

where $K = K(q, \dot{q}, \delta q)$ is some homogeneous function of $\{\delta q^\mu\}$. Hence

$$\delta L = -\frac{\mathrm{d}K}{\mathrm{d}t} - \delta H. \tag{2.8}$$

Taking into account the operator equation (2.3), we can rewrite (2.6)–(2.8) in the following form:

$$\delta L = -\frac{\mathrm{d}K}{\mathrm{d}t} - \frac{1}{\mathrm{i}\hbar}[H, G] = -\frac{\mathrm{d}K}{\mathrm{d}t} + \frac{\mathrm{d}G}{\mathrm{d}t}. \tag{2.9}$$

On the other hand, the variation of L under the transformation (2.5) can be expressed in a standard way (by extracting a total time derivative) by

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^\mu} \star \delta q^\mu\right) + \frac{\delta L}{\delta q^\mu} \star \delta q^\mu, \tag{2.10}$$

where the objects

$$\frac{\partial L}{\partial \dot{q}^\mu} \star \delta q^\mu := \mathcal{P}(q, \dot{q}, \delta q), \quad \frac{\delta L}{\delta q^\mu} := \mathcal{E}(q, \dot{q}, \delta q) \tag{2.11}$$

denote homogeneous δq functions. The symbol "$\star$" is used for the sake of clarity; in the classical limit $\hbar \to 0$ the combinations (2.11) become the product of the classical moment with δq and the contraction of the Euler–Lagrange equations with δq, respectively[2].

Comparing (2.9) with (2.10) we obtain

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(K + \frac{\partial L}{\partial \dot{q}^\mu} \star \delta q^\mu\right) + \frac{\delta L}{\delta q^\mu} \star \delta q^\mu. \tag{2.12}$$

Since (2.12) must agree with (2.2) we can write

$$G = K + \frac{\partial L}{\partial \dot{q}^\mu} \star \delta q^\mu, \quad \frac{\delta L}{\delta q^\mu} \star \delta q^\mu = 0. \tag{2.13}$$

The first expression in (2.13) gives the definition of the generator G, the second one is related to the dynamical equations for the operators q^μ in the Euler–Lagrange form.

Further development of the theory requires the explicit form of the Lagrangian and investigation of the transformation properties of the operators.

[1] Permissible variations are called *elementary* (or c-number) ones if they commute with all the operators of a model. An elementary variation is equivalent to the unit operator. In [14] Schwinger has considered the model with elementary variations as permissible ones

[2] When $f = f(q)$ and $[q^\mu, \dot{q}^\nu] \neq 0$, one cannot write down the variation δf in the explicit form in the general case. To demonstrate what the symbol "$\star$" means, let us consider the simplest case: $f(q) = q^1 q^2 q^3$. Then

$$\delta f(x) = \delta q^1 q^2 q^3 + q^1 \delta q^2 q^3 + q^1 q^2 \delta q^3 := \frac{\partial f}{\partial q^\mu} \star \delta q^\mu,$$

where

$$\frac{\partial f}{\partial q^1} = q^2 q^3, \quad \frac{\partial f}{\partial q^2} = q^1 q^3, \quad \frac{\partial f}{\partial q^3} = q^1 q^2$$

denote "derivatives", obtained by omitting the factors q^1, q^2, q^3, respectively. To write down the variation δf one has to insert δq^μ into $\partial f / \partial q^\mu$, as we can see from these formulae

3 Transformation properties of operators

The construction of the quantum mechanics for the present model is essentially based on the assumption that the coordinate operators $\{q^\mu : \mu = \overline{1,n}\}$ form a complete set of commuting observables. If the complete set is replaced with another one consistent with the initial set (i.e. the operators of the new set are functions of the operators from the old one), the physical meaning of the theory does not change. In particular, when the complete set is identified with the coordinate operators $\{q^\mu\}$, such a change of the complete set is nothing but the coordinate transformation $q^\mu \to \overline{q}^\mu := \overline{q}^\mu(q)$. All the geometrical objects, which are functions of only the $\{q^\mu\}$'s commute with them.

In this paper we assume $[q^\mu, \dot{q}^\nu]$ to be a function of only the $\{q^\mu\}$'s. This assumption can be motivated as follows. The model has to be invariant under an arbitrary (smooth) coordinate transformation $q \to \overline{q} = \overline{q}(q)$, which is associated with the change of the set of commuting observables. To obtain the Lagrangian $L(q, \dot{q})$ in the new coordinate system, we have to make the substitution $q = q(\overline{q})$, $\dot{q} = \dot{q}(q, \dot{\overline{q}})$, where $\dot{\overline{q}} = \mathrm{d}[\overline{q}(q)]/\mathrm{d}t$. The important expression $\dot{\overline{q}}^\mu = \dot{q}^\alpha \circ \partial_\alpha \overline{q}^\mu(q)$ holds if and only if $[q^\mu, q^\nu] = 0$ and $[q^\mu, \dot{q}^\nu] = \{\text{some function of } q\}$, see the Appendix A (here the symbol "$\circ$" denotes the symmetrized Jordan product[3]).

Therefore, this assumption allows us to analyze the general situation, where the form of the metric tensor and coordinate transformation are not given explicitly. Of course, after determination of the commutation relations, one has to verify its correspondence with the basic assumptions.

The transformation rule of a geometrical object under a point coordinate transformation $q^\mu \to \overline{q}^\mu := \overline{q}^\mu(q)$ depends on its operator properties and inner structure. In this section we extend the classical definitions of a scalar, a vector and a tensor to the case of non-commutative operators.

A *quantum scalar* we treat as a quantum geometrical object, which does not change value under the coordinate transformation

$$\overline{f}(\overline{q}) = f(q) \tag{3.1}$$

(here the argument in brackets points to the method of description, not to the functional dependence in general).

A *quantum vector* is a quantum geometrical object with one index which transforms under the coordinate transformation by one of the following rules:

(1) A^μ (A_μ) is a *left-side vector*, if

$$\overline{A^\mu}(\overline{q}) = \overline{a}^\mu_\nu(q) A^\nu(q) \quad \left(\overline{A_\mu}(\overline{q}) = a^\nu_\mu(q) A_\nu(q)\right); \tag{3.2}$$

(2) A^μ (A_μ) is a *right-side vector*, if

$$\overline{A^\mu}(\overline{q}) = A^\nu(q) \overline{a}^\mu_\nu(q) \quad \left(\overline{A_\mu}(\overline{q}) = A_\nu(q) a^\nu_\mu(q)\right); \tag{3.3}$$

(3) A^μ (A_μ) is a *two-side vector*, if it is a left- and right-side vector simultaneously:

$$\begin{aligned} \overline{A}^\mu(\overline{q}) &= A^\nu(q)\overline{a}^\mu_\nu(q) = \overline{a}^\mu_\nu(q) A^\nu(q), \\ \overline{A}_\mu(\overline{q}) &= A_\nu(q) a^\nu_\mu(q) = a^\nu_\mu(q) A_\nu(q). \end{aligned} \tag{3.4}$$

(4) A^μ (A_μ) is a *symmetrized vector*, if

$$\overline{A^\mu}(\overline{q}) = \overline{a}^\mu_\nu(q) \circ A^\nu(q) \quad \left(\overline{A_\mu}(\overline{q}) = a^\nu_\mu(q) \circ A_\nu(q)\right), \tag{3.5}$$

with the following notation:

$$\overline{a}^\mu_\nu = \frac{\partial \overline{q}^\mu}{\partial q^\nu}, \quad a^\mu_\nu = \frac{\partial q^\mu}{\partial \overline{q}^\nu} \tag{3.6}$$

for the transformation matrices.

The transformation laws described by (3.2) and (3.3) correspond to non-hermitean operators. As a two-side vector we can consider an arbitrary vector which depends on only the coordinate operators $\{q^\mu\}$ (and commutes with the transformation matrices (3.6)). Finally, the transformation laws described by (3.5) correspond to hermitean operators. For example, the operator of a generalized velocity $\dot{q}^\mu$ transforms as a symmetrized vector

$$\dot{q}^\mu \to \dot{\overline{q}}^\mu = \overline{a}^\mu_\nu \circ \dot{q}^\nu \tag{3.7}$$

(we have taken into account the fact that $[q^\mu, \dot{q}^\nu]$ is a function of only the $\{q^\mu\}$'s). Every symmetrized vector can be expressed as the sum of left and right parts (see Sect. 5).

In the remaining part of this section we summarize some useful properties of quantum geometrical objects. Let us prove that the contraction of symmetrized and two-side tensors is a symmetrized tensor. We define the operator

$$p_\mu := g_{\mu\nu} \circ \dot{q}^\nu, \tag{3.8}$$

where $g_{\mu\nu} = g_{\nu\mu}$ is a two-side tensor. In new coordinates this operator receives the form

$$\begin{aligned} \overline{p}_\mu &= \overline{g}_{\mu\nu} \circ \dot{\overline{q}}^\nu = \overline{g}_{\mu\nu} \circ (\overline{a}^\nu_\alpha \circ \dot{q}^\alpha) = \left(\overline{g}_{\mu\nu} \circ \overline{a}^\nu_\alpha\right) \circ \dot{q}^\alpha \\ &= \left(g_{\alpha\nu} \circ a^\nu_\mu\right) \circ \dot{q}^\alpha = a^\nu_\alpha \circ (g_{\alpha\nu} \circ \dot{q}^\alpha) = a^\nu_\mu \circ p_\nu. \end{aligned} \tag{3.9}$$

In a similar way one can demonstrate that

$$\dot{q}^\mu = g^{\mu\nu} \circ p_\nu \to \dot{\overline{q}}^\mu = \overline{a}^\mu_\nu \circ \dot{q}^\nu. \tag{3.10}$$

When $\{g_{\mu\nu}\}$ and $\{g^{\mu\nu}\}$ have the sense of covariant and contravariant metric tensors respectively, (3.8)–(3.10) can be interpreted as a rule for lowering and raising of the index in quantum (symmetrized) tensors.

Similarly, the Jordan contraction (or a scalar product) of two-side and symmetrized vectors is a quantum scalar. To prove this, let us consider the symmetrized vector p_μ and the two-side vector v^μ. Then

$$\begin{aligned} \overline{p}_\mu \circ \overline{v}^\mu &= \overline{v}^\mu \circ \left(a^\alpha_\mu \circ p_\alpha\right) \\ &= p_\alpha \circ \left(v^\mu \circ a^\alpha_\mu\right) = p_\alpha \circ v^\alpha. \end{aligned} \tag{3.11}$$

[3] This product is defined by $a \circ b := (1/2)(ab + ba)$ for arbitrary operators a and b. See the Appendix A for its properties

As to the Jordan contraction of symmetrized vectors, its properties are more complicated than previously mentioned. Generally, such objects are not quantum tensors and to determine them one has to use the explicit form of the commutation relations.

Now we consider the transformation properties of the commutation relations:

(1) for a commutator between a quantum scalar $f =: f(q)$ and a quantum vector p_μ we have

$$\left[\overline{f}, \overline{p}_\mu\right] = \left[f, a^n_\mu \circ p_\nu\right] = a^n_\mu \circ \left[f, p_\nu\right],$$

where $[f, a^\mu_\nu] = 0$ is assumed. According to (3.1)–(3.5), this object is a quantum vector;

(2) a commutator between a two-side symmetrized vector and a quantum vector p_μ reads

$$\begin{aligned}[\overline{v}^\mu, \overline{p}_\nu] &= \left[\overline{a}^\mu_\alpha v^\alpha, a^\beta_\nu \circ p_\beta\right] = a^\beta_\nu \left[\overline{a}^\mu_\alpha v^\alpha, p_\beta\right] \\ &= a^\beta_\nu \circ \left(\overline{a}^\mu_\alpha \circ [v^\alpha, p_\beta] + v^\alpha \circ [\overline{a}^\mu_\alpha, p_\beta]\right).\end{aligned}$$

The second term in the right hand side does not have a tensor meaning, therefore $[v^\mu, p_\nu]$ is not a quantum tensor;

(3) for the commutator between two symmetrized vectors we write

$$\begin{aligned}\left[\overline{A}^\mu, \overline{B}^\nu\right] &= \left[\overline{a}^\mu_\alpha \circ A^\alpha, \overline{a}^\nu_\beta \circ B^\beta\right] \\ &= \overline{a}^\mu_\alpha \overline{a}^\nu_\beta \circ [A^\alpha, B^\beta] + \overline{a}^\mu_\alpha \circ \left([A^\alpha, \overline{a}^\nu_\beta] \circ B^\beta\right) \\ &\quad + \left(\overline{a}^\nu_\beta \circ \left[\overline{a}^\mu_\alpha, B^\beta\right]\right) \circ A^\alpha.\end{aligned}$$

Analogously to previously, $[A^\mu, B^\nu]$ fails to be a quantum tensor due to the presence of two non-covariant tensor terms at the right hand side.

In connection with the previously considered transformation properties of quantum geometrical objects one can observe that there are some fundamental complications in the formulation of quantum mechanics on a Riemannian manifold.

Namely, the naive definition of the quantum Lagrangian for a free particle in a curved space based on the classical expression

$$L_0 = \frac{1}{2}\dot{q}^\mu g_{\mu\nu}(q)\dot{q}^\nu \tag{3.12}$$

leads one to the conclusion that the Lagrangian, being a hermitean operator, fails to be a quantum scalar (in the sense of the definition (3.1)), i.e. $\overline{L}_0(\overline{q}) \neq L_0(q)$. In explicit form we write

$$\begin{aligned}\overline{L}_0 &= \frac{1}{2}\dot{\overline{q}}^\mu \overline{g}_{\mu\nu}\dot{\overline{q}}^\nu \\ &= L_0 + \frac{1}{4}\left[\dot{q}^\alpha, g_{\alpha\beta}a^\beta_\mu b^\mu\right] - \frac{1}{8}a^\alpha_\mu a^\beta_\nu g_{\alpha\beta} b^\mu b^\nu,\end{aligned} \tag{3.13}$$

where $b^\mu := b^\mu(q) := [a^\mu_\alpha(q), \dot{q}^\alpha]$. Therefore, the main requirement for the Lagrangian (scalar invariance) is violated. Note that in the classical limit the second and third terms at the right hand side of (3.13) vanish, being functions of the second (or higher) order of $\hbar$.

4 Quantum Lagrangian for free particle in curved space

Now let us concentrate our attention on the formulation of the quantum mechanics for a free particle in a Riemannian space equipped with the metric $g_{\mu\nu}(q)$. The first step in this formulation consists of the construction of the quantum Lagrangian which is invariant under the coordinate transformation $q \to \overline{q} = \overline{q}(q)$. We introduce the following form of the Lagrangian:

$$L = \frac{1}{2}\dot{q}^\mu g_{\mu\nu}(q)\dot{q}^\nu - U_q(q), \tag{4.1}$$

where $U_q = U_q(q)$ is some function which permits the function (4.1) to be a quantum scalar under a point coordinate transformation (U_q may be conditionally called a "quantum potential"[4]). Its explicit form is unknown at this stage, because to determine it we have to use commutation relations.

For the operators of the quantum mechanics described by (4.1) we make the following basic assumptions:

(1) $[q^\mu, q^\nu] = 0$ and $\{q^\mu : \mu = \overline{1, n}\}$ form a complete set of commuting observables;

(2) $[q^\mu, \dot{q}^\nu]$ is a function of only the $\{q^\mu\}$'s.

These assumptions permit us to conclude that U_q is a function of only $\{q^\mu\}$'s.

Taking the total time derivative of $[q^\mu, q^\nu] = 0$ we can see that

$$\frac{1}{i\hbar}[q^\mu, \dot{q}^\nu] = f^{\mu\nu}(q) = f^{\nu\mu}(q).$$

Note that the ordering of factors in (4.1), which satisfies the hermitean condition, is not unique. We can take the expression $\dot{q}^\mu \circ (g_{\mu\nu} \circ \dot{q}^\nu)$ instead of $\dot{q}^\mu g_{\mu\nu}\dot{q}^\nu$. Such a replacement leads to another function $U_q(q)$. The scalar Lagrangian L is the same (one can demonstrate this fact after determination of the commutation relations). Now we consider the transformation properties of $U_q(q)$ and $L(q, \dot{q})$ under the infinitesimal coordinate transformation $q^\mu \to q^\mu + \delta q^\mu(q)$. Taking into account the fact that under such a transformation the velocity operator and the metric change as

$$\delta_c \dot{q}^\mu = \dot{q}^\alpha \circ \partial_\alpha \delta q^\mu, \quad \delta_c g_{\mu\nu} = -g_{\mu\alpha}\partial_\nu \delta q^\alpha - g_{\alpha\nu}\partial_\mu \delta q^\alpha,$$

we can write

$$\delta_c L_0 = \frac{1}{4}\left[\left[\partial_\alpha \delta q^\mu, \dot{q}^\alpha\right] g_{\mu\nu}, \dot{q}^\nu\right], \tag{4.2}$$

where we have used the basic assumptions.

Because the Lagrangian $L = L_0 - U_q$ is a scalar, we have $\delta_c L = 0$; then

$$\delta_c U_q = \delta_c L_0 = \frac{1}{4}\left[\left[\partial_\alpha \delta q^\mu, \dot{q}^\alpha\right] g_{\mu\nu}, \dot{q}^\nu\right]. \tag{4.3}$$

[4] The term "quantum potential" is used in several works with a somewhat different meaning [1]. Therefore we set this term in quotation marks

This formula shows that neither L_0 nor U_q is a quantum scalar; the combination $L = L_0 - U_q$ has this property.

Further, let us write down the variation of the operator $L(q, \dot{q})$ under the alteration of its arguments:

$$\delta L(q, \dot{q}) := L(q + \delta q, \dot{q} + \delta\dot{q}) - L(q, \dot{q}). \qquad (4.4)$$

If the variation "δ" satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta q^\mu = \delta\dot{q}^\mu$$

and

$$\delta g_{\mu\nu} = g_{\mu\nu}(q + \delta q) - g_{\mu\nu}(q) = \delta q^\alpha \partial_\alpha g_{\mu\nu},$$

we obtain

$$\begin{aligned}\delta L(q, \dot{q}) = &\frac{1}{2}\dot{q}^\mu(\delta q^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu \delta q^\alpha + g\_\alpha\nu \partial_\mu \delta q^\alpha) q^\nu \\ &+ \frac{1}{4}\left[\left[\partial_\alpha \delta q^\mu, \dot{q}^\alpha\right] g_{\mu\nu}, \dot{q}^\nu\right] - \delta q^\alpha \partial_\alpha U_q(q). \qquad (4.5)\end{aligned}$$

The change of the form of the geometrical object $F(q)$ under the transformation $q \to q + \delta q$ is defined by

$$\delta_0 F(q) = \delta_c F(q) - \delta q^\mu \partial_\mu F(q)$$

(when $F(q)$ is a tensor object, the variation "δ_0" is called the *Lie variation*). In particular, the variation of the form of the non-scalar function $U_q(q)$ has the form

$$\delta_0 U_q = \frac{1}{4}\left[\left[\partial_\alpha \delta q^\mu, \dot{q}^\alpha\right] g_{\mu\nu}, \dot{q}^\nu\right] - \delta q^\alpha \partial_\alpha U_q(q).$$

Hence we can rewrite (4.5) as

$$\delta L = -\frac{1}{2}\dot{q}^\mu g_{\mu\nu}\dot{q}^\nu + \delta_0 U_q. \qquad (4.6)$$

The variation $\delta(\ldots)$ is permissible, if δL reduces to the total time derivation of some function (see Sect. 2). Without loss of generality we can assume this for δL.

Comparing the factors corresponding to different powers of $\dot{q}$, we can find that

$$\begin{aligned}&\delta_L g_{\mu\nu} = 0, \\ &\delta_0 U_q = \frac{1}{4}\left[\left[\partial_\alpha \delta q^\mu, \dot{q}^\alpha\right] g_{\mu\nu}, \dot{q}^\nu\right] - \delta q^\mu \partial_\mu U_q = 0. \qquad (4.7)\end{aligned}$$

The first equation in (4.7) means that $\{\delta q^\mu\}$ is a Killing vector for the metric $\{g_{\mu\nu}\}$. Every solution of the Killing equation $\delta_0 g_{\mu\nu} = 0$ can be decomposed as $\delta q^\mu = \varepsilon^a v^\mu_a$, where $\varepsilon^a = \text{const}$ (an infinitesimal c-number) and $\{v^\mu_a : \mu = \overline{1,n},\ a = \overline{1,m}\}$ are m independent solutions of this equation.

Comparing (4.6) with the general expression of the variation of the Lagrangian we find that $\delta H = 0$ (for unknown H). As a consequence, the generator of permissible variations $\delta q^\mu = \varepsilon^a v^\mu_a$ takes the form

$$\begin{aligned}&G = p_\mu \circ \delta q^\mu = (p_\mu \circ v^\mu_a)\,\varepsilon^a := \varepsilon^a p_a, \\ &p_a := p_\mu \circ v^\mu_a, \quad p_\mu = g_{\mu\nu} \circ \dot{q}^\nu, \qquad (4.8)\end{aligned}$$

where the permissible variations $\delta q^\mu = \varepsilon^a v^\mu_a$ are Killing vectors expressed as the linear combination of the independent solutions $\{v_a\}$ of the equations $\delta_0 g_{\mu\nu} = 0$.

Therefore, we conclude that the features of quantum mechanics on a curved space V_n essentially depend on the properties of its group of isometries. This group appears in the theory in the generator $G(t)$. The group properties of $\{v^\mu_a\}$ are expressed by

$$v^\mu_a \partial_\mu v^\nu_b - v^\mu_b \partial_\mu v^\nu_a = c^c{}_{ab} v^\nu_c, \qquad (4.9)$$

where $c^a{}_{bc}$ are the structure constants of a group. The set of vector fields $\{v^\mu_a \partial_\mu\}$ forms the representation of the Lie algebra induced by the representation of the Lie group of isometries.

On the other hand, the variation δL can be rewritten as

$$\begin{aligned}\delta L = &\frac{\mathrm{d}}{\mathrm{d}t}(G) - \dot{p}^\mu \circ \delta q^\mu + \frac{1}{2}\dot{q}^\mu(\delta q^\alpha \partial_\alpha g_{\mu\nu})\dot{q}^\nu \\ &- \delta q^\mu \partial_\mu U_q + \frac{1}{2}[\delta q^\mu, \dot{g}_\mu], \qquad (4.10)\end{aligned}$$

where

$$g_\mu = \frac{1}{2}[\dot{q}^\nu, g_{\mu\nu}].$$

When the symbol $\delta(\ldots)$ corresponds to the permissible variations this equation falls in two equations: the first one describes the conservation of the generator,

$$\frac{\mathrm{d}}{\mathrm{d}t}G = 0,$$

and the second one contains the equations of motion:

$$\begin{aligned}\dot{p}_\mu \circ \delta q^\mu = &\frac{1}{2}\dot{q}^\mu(\partial_\alpha g_{\mu\nu}\delta q^\alpha)\dot{q}^\nu \qquad (4.11) \\ &- \delta q^\mu \partial_\mu U_q + \frac{1}{2}[\delta q^\mu, \dot{g}_\mu].\end{aligned}$$

In order to eliminate δq^μ we have to use the canonical commutation relations (unknown at this stage).

At the end of this section we point out the fact that then for an arbitrary operator A we can write

$$\begin{aligned}\delta\frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{\mathrm{i}\hbar}\left[\frac{\mathrm{d}A}{\mathrm{d}t}, G\right] \\ &= \frac{1}{\mathrm{i}\hbar}\frac{\mathrm{d}}{\mathrm{d}t}[A, G] - \frac{1}{\mathrm{i}\hbar}\left[A, \frac{\mathrm{d}G}{\mathrm{d}t}\right] = \frac{\mathrm{d}\delta A}{\mathrm{d}t}, \qquad (4.12)\end{aligned}$$

i.e. the total time derivation commutes with the operation "δ" of permissible variations.

5 Commutation relations in case of simply transitive group

In the present paper we will consider the simplest case of quantum mechanics on a curved space. Namely, we restrict the group of isometries of V_n to be a simply transitive transformation group on V_n. Such a simplification permits the commutation relations to be directly determined from (2.3).

It is important to point out here that due to $U_q = U_q(q)$, this function does not make any contribution to the generator. Therefore, the explicit form of U_q is not required in this section. Moreover, it can be calculated when the commutation relations are given.

In the case of the simply transitive group acting on V_n the number of independent Killing vectors equals the dimension of V_n, i.e. $m = n$ and $\mathrm{Rg}(v^\mu_a) = n$. Then we can introduce the inverse of $\{v^\mu_a\}$ by

$$e^a_\mu v^\nu_a = \delta^\nu_\mu, \quad e^\alpha_\mu v^\mu_b = \delta^a_b,$$

which obeys the Maurer–Cartan equation

$$\partial_\mu e^a_\nu - \partial_\nu e^a_\mu = -c^a{}_{bc} e^b_\mu e^c_\nu.$$

To determine the commutation relations we use (2.3). The symbol δ corresponding to the permissible variations in our case means the shift

$$\delta F(q, \dot{q}) = F(q + \delta q, \dot{q} + \delta\dot{q}) - F(q, \dot{q}),$$

for any operator $F(q, \dot{q})$. Such a variation corresponds to a unitary transformation that acts on different objects equally.

At first let us employ this relation for coordinate operators:

$$\delta q^\mu = \frac{1}{\mathrm{i}\hbar}\left[q^\mu, G\right]. \tag{5.1}$$

Using the explicit form of the generator of permissible variations (4.8) and $\delta q^\mu = \varepsilon^a v^\mu_a$ due to the arbitrariness of the c-number parameters $\{\varepsilon^a\}$ we obtain from (5.1)

$$\left(\delta^\mu_\nu - \frac{1}{\mathrm{i}\hbar}[q^\mu, p_\nu]\right) \circ v^\nu_a = 0. \tag{5.2}$$

Multiplying (5.2) by the inverse matrix $\{e^a_\alpha\}$ we obtain the following commutation relation:

$$[q^\mu, p_\nu] = \mathrm{i}\hbar\delta^\mu_\nu. \tag{5.3}$$

Further let us consider the operator equation

$$\delta p_\mu = \frac{1}{\mathrm{i}\hbar}\left[p_\mu, G\right]. \tag{5.4}$$

Under the transformation $q \to q + \delta q(q)$ the symmetrized vector p_μ changes as

$$\delta p_\mu = -\frac{\partial \delta q^\nu}{\partial q^\mu} \circ p_\nu.$$

By making use of this relation and (5.4), one easily finds

$$[p_\mu, p_\nu] \circ v^\nu_a = 0. \tag{5.5}$$

Multiplying (5.5) by the inverse matrix $\{e^a_\mu\}$ we have

$$[p_\mu, p_\nu] = 0. \tag{5.6}$$

Therefore, the commutation relations (5.3) and (5.6) correspond to canonical commutation relations for the canonical momentum p_μ conjugate to the coordinate q^μ. The other commutation relations of the theory can be calculated using (5.3) and (5.6). In particular, (5.4) is equivalent to

$$[q^\mu, \dot{q}^\nu] = \mathrm{i}\hbar g^{\mu\nu}. \tag{5.7}$$

We can observe that the basic assumption about the commutator $[q^\mu, \dot{q}^\nu]$ is in accordance with (6.7). For an arbitrary two-side geometrical object $F = F(q)$ we have

$$[F, p_\mu] = \mathrm{i}\hbar\partial_\mu F. \tag{5.8}$$

For example, let us consider the commutation relation between the current operators $p_a := p_\mu \circ v^\mu_a$ (which are quantum scalars according to (3.11)). A direct calculation using (5.3), (5.6) and (4.9) leads to

$$[p_a, p_b] = -\mathrm{i}\hbar c^c{}_{ab} p_c. \tag{5.9}$$

Now we discuss the transformation properties of the commutation relations under a coordinate transformation. Obviously, the following commutation relation can be carried out for any two-side geometrical object $F = F^A(q)$, where $A = \{\alpha_1, \ldots \alpha_k\}$ is a multiindex:

$$\left[F^A, p_\mu\right] = \mathrm{i}\hbar\partial_\mu F^A. \tag{5.10}$$

In new coordinates F^A and p_μ take the form

$$\overline{p}_\mu = a^\nu_\mu \circ p_\nu, \quad \overline{F}^A = \overline{a}^A_B F^B, \tag{5.11}$$

where

$$\left[F^A, a^\mu_\nu\right] = 0, \quad \overline{a}^A_B = \overline{a}^{\alpha_1}_{\beta_1} \ldots \overline{a}^{\alpha_k}_{\beta_k}.$$

Then

$$\begin{aligned}\left[\overline{F}^A, \overline{p}_\mu\right] &= \left[\overline{a}^A_B F^B, a^\nu_\mu \circ p_\nu\right] \\ &= \mathrm{i}\hbar a^\nu_\mu \partial_\nu \left(\overline{a}^A_B F^B\right) = \mathrm{i}\hbar\overline{\partial}_\mu \overline{F}^A.\end{aligned} \tag{5.12}$$

This result shows that all commutation relations are form invariant under a general coordinate transformation. Therefore, the procedure of the determination of the commutation relations is self-consistent.

6 Determination of quantum correction

To derive the quantum Lagrangian of a free particle in a curved space, it is necessary to find the quantum correction U_q. Our procedure for its determination is based on the construction of the invariant norm of a quantum vector.

In the case of a two-side vector $\{A^\mu\}$ the norm has the following form:

$$(A, A) = \|A\|^2 := A_\mu g^{\mu\nu} A_\nu = A^\mu g_{\mu\nu} A^\nu. \tag{6.1}$$

If the quantum vector $\{A^\mu\}$ does not commute with q^μ, the expression (6.1) fails to be a quantum scalar according to Sect. 4.

To extend the expression (6.1) for the case of a symmetrized vector, let us consider the transformation law of p_μ under a general coordinate transformation:

$$\overline{p}_\mu = a^\nu_\mu \circ p_\nu = a^\nu_\mu p_\nu - \frac{\mathrm{i}\hbar}{2}\partial_\nu a^\nu_\mu, \tag{6.2}$$

or

$$\overline{p}_\mu = a^\nu_\mu \circ p_\nu = p_\nu a^\nu_\mu + \frac{\mathrm{i}\hbar}{2}\partial_\nu a^\nu_\mu. \tag{6.3}$$

The contracted Christoffel symbol $\Gamma_\mu := \Gamma^\alpha_{\mu\alpha}$ transforms as

$$\overline{\Gamma}_\mu = a^\nu_\mu \Gamma_\nu + \partial_\nu a^\nu_\mu. \tag{6.4}$$

We define two non-hermitean operators:

$$\pi_\mu := p_\mu + \frac{\mathrm{i}\hbar}{2}\Gamma_\mu, \quad \pi^\dagger_\mu := p_\mu - \frac{\mathrm{i}\hbar}{2}\Gamma_\mu, \tag{6.5}$$

with the following properties:

$$\begin{aligned} p_\mu &= \frac{1}{2}(\pi_\mu + \pi^\dagger_\mu), \quad (\pi_\mu)^\dagger = \pi^\dagger_\mu, \quad (\pi^\dagger_\mu)^\dagger = \pi_\mu, \\ [\pi_\mu, \pi_\nu] &= 0, \quad [\pi^\dagger_\mu, \pi^\dagger_\nu] = 0, \\ [\pi_\mu, \pi^\dagger_\nu] &= \mathrm{i}\hbar\partial_\mu\Gamma_\nu = \mathrm{i}\hbar\partial_\mu\Gamma_\nu. \end{aligned} \tag{6.6}$$

Using (6.4)–(6.6) one can observe that π_μ and $\pi^\dagger_\mu$ behave under the coordinate transformation as left-side and right-side quantum vectors, respectively:

$$\overline{\pi}_\mu = a^\nu_\mu \pi_\nu, \quad \overline{\pi}^\dagger_\mu = \pi^\dagger_\mu a^\nu_\mu. \tag{6.7}$$

Taking into account the transformation laws of π_μ and $\pi^\dagger_\mu$ we introduce the quantum norm of the symmetrized vector p_μ by

$$(p,p) = \|p\|^2 := \pi^\dagger_\mu g^{\mu\nu}\pi_\nu. \tag{6.8}$$

Similarly, one can define two non-hermitean operators connected with $\dot{q}^\mu$:

$$V^\mu := \dot{q}^\mu - \frac{\mathrm{i}\hbar}{2}\Phi^\mu, \quad V^{\mu\dagger} := \dot{q}^\mu + \frac{\mathrm{i}\hbar}{2}\Phi^\mu, \tag{6.9}$$

with the following transformation properties:

$$\overline{V}^\mu = \overline{a}^\mu_\nu V^\nu, \quad \overline{V}^{\mu\dagger} = V^{\nu\dagger}\overline{a}^\mu_\nu, \tag{6.10}$$

where $\Phi^\mu := g^{\alpha\beta}\Gamma^\mu{}_{\alpha\beta}$.

Due to (6.10) we introduce the quantum norm of the symmetrized vector $\dot{q}^\mu$ by

$$(\dot{q},\dot{q}) = \|\dot{q}\|^2 = V^{\mu\dagger}g_{\mu\nu}V^\nu. \tag{6.11}$$

Further, taking into account the connection $p_\mu = g_{\mu\nu} \circ \dot{q}^\nu$ one can directly prove

$$(p,p) = (\dot{q},\dot{q}), \tag{6.12}$$

i.e. the quantum norms of the velocity and the momentum operators introduced above have the same value, as they must.

These properties lead us to write the Lagrangian in the following form:

$$L = \frac{1}{2}(\dot{q},\dot{q}). \tag{6.13}$$

Rewriting this relation using (6.9) and (6.10) we find the explicit form of U_q:

$$\begin{aligned} U_q = &-\frac{\hbar^2}{4}\left(\partial_\mu\Gamma^\mu + \frac{1}{2}\Gamma_\mu\Gamma^\mu\right) \\ &-\frac{\hbar^2}{4}\left(\partial_\mu\Theta^\mu - \frac{1}{2}\Theta_\mu\Theta^\mu\right), \end{aligned} \tag{6.14}$$

where

$$\Gamma^\mu = g^{\mu\nu}\Gamma_\nu, \quad \Theta^\mu := \partial_\nu g^{\mu\nu}, \quad \Theta_\mu = g_{\mu\nu}\Theta^\nu, \tag{6.15}$$

and

$$\Gamma^\mu + \Theta^\mu + \Phi^\mu = 0.$$

We also can write (6.14) as

$$U_q = \frac{\hbar^2}{4}\left(\partial_\mu\Phi^\mu + \frac{1}{2}g_{\mu\nu}\Phi^\mu\Phi^\nu + \Gamma_\mu\Phi^\mu\right). \tag{6.16}$$

Using the commutation relations it is easy to derive the useful identity

$$p_\mu g^{\mu\nu}p_\nu = \dot{q}^\alpha g_{\alpha\beta}\dot{q}^\beta + \frac{\hbar^2}{2}\left(\partial_\mu\Theta^\mu - \frac{1}{2}\Theta_\mu\Theta^\mu\right). \tag{6.17}$$

Taking into account (6.17) one can rewrite (6.13) as

$$L = \frac{1}{2}p_\mu g^{\mu\nu}p_\nu + \frac{\hbar^2}{4}\left(\partial_\mu\Gamma^\mu + \frac{1}{2}\Gamma_\mu\Gamma^\mu\right). \tag{6.18}$$

The form of the Lagrangian (6.18) as far as the expression for the norm of the quantum vector $\dot{q}$ is concerned is fixed by the commutation relations. The choice of the *definitions* (6.11) and (6.18) is motivated by their similarity to the classical expressions that contain geometrical objects, derived from the metric $g_{\mu\nu}$ and operators q, $\dot{q}$.

In order to construct the Hamiltonian of a free particle in a curved space we consider the quantum version of the Legendre transformation:

$$H = p_\mu \star \dot{q}^\mu - L, \tag{6.19}$$

where

$$p_\mu \star \dot{q}^\mu := \frac{1}{2}\left(\pi^\dagger_\mu V^\mu + V^{\mu\dagger}\pi_\mu\right) = (p,p) \tag{6.20}$$

is a scalar.

From (6.19) and (6.20) we directly obtain

$$H = (p,p) - L = \frac{1}{2}(p,p) = L. \tag{6.21}$$

So our Lagrangian is purely kinetic.

The Hamiltonian can be rewritten in another form, which is similar to the classical one:

$$H = p_\mu \circ \dot{q}^\mu - L - Z, \tag{6.22}$$

where Z is an auxiliary variable introduced by Sugano [1]:

$$Z = -\frac{\hbar^2}{4} R + \frac{\hbar^2}{4} g^{\alpha\beta} \Gamma^\mu{}_{\nu\alpha} \Gamma^\nu{}_{\mu\beta}. \tag{6.23}$$

Here R is the scalar curvature of V_n.

As a result, we have defined all the objects appearing in quantum mechanics with the simply transitive transformation group of isometries.

7 Equations of motion

Now we rewrite the form of the Euler–Lagrange equations obtained in Sect. 5 (see (4.11)) as the following variational equation:

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t}(p_\mu \circ \delta q^\mu) - \dot{p}^\mu \circ \delta q^\mu + \frac{1}{2}\dot{q}^\mu(\delta q^\alpha \partial_\alpha g_{\mu\nu})\dot{q}^\nu$$
$$- \delta q^\mu \partial_\mu U_q + \frac{1}{2}[\delta q^\mu, \dot{g}_\mu], \tag{7.1}$$

where

$$g_\mu = \frac{1}{2}[\dot{q}^\nu, g_{\mu\nu}].$$

From (7.1) one can draw the conclusion that the generator of canonical variations is $G = p_\mu \circ \delta q^\mu$ and the equations of motion are contained in the relation

$$\dot{p}_\mu \circ \delta q^\mu = \frac{1}{2}\dot{q}^\mu(\partial_\alpha g_{\mu\nu} \delta q^\alpha)\dot{q}^\nu$$
$$- \delta q^\mu \partial_\mu U_q + \frac{1}{2}[\delta q^\mu, \dot{g}_\mu]. \tag{7.2}$$

Using the set of commutation relations obtained above one can transform (7.2) to the form

$$\dot{p}_\mu \circ \delta q^\mu = f_\mu \circ \delta q^\mu + T^{\mu\nu}\delta_0 g_{\mu\nu}, \tag{7.3}$$

where

$$f_\mu = -\frac{1}{2} p_\alpha \partial_\mu g^{\alpha\beta} p_\beta - \frac{\hbar^2}{4}\partial_\mu\left(\partial_\alpha \Gamma^\alpha + \frac{1}{2}\Gamma_\alpha \Gamma^\alpha\right), \tag{7.4}$$

where $T^{\mu\nu} = T^{\mu\nu}(q) \sim \hbar^2$ is some tensor of the second rank. The variation δq^μ is a Killing vector; therefore the second term in (7.3) vanishes due to the condition $\delta_0 g_{\mu\nu} = 0$. Using the decomposition $\delta q^\mu = \varepsilon^a v_a^\mu$ we can write (7.3) as

$$(\dot{p}_\mu - f_\mu) \circ v_a^\mu = 0. \tag{7.5}$$

As far as the matrix $\{v_a^\mu\}$ is invertible and describes a two-side vector, we can eliminate it from (7.5) by multiplication by its inverse $\{e_\mu^a\}$. Finally, we obtain the equation of motion in the following form:

$$\dot{p}_\mu = -\frac{1}{2} p_\alpha \partial_\mu g^{\alpha\beta} p_\beta - \frac{\hbar^2}{4}\partial_\mu\left(\partial_\alpha \Gamma^\alpha + \frac{1}{2}\Gamma_\alpha \Gamma^\alpha\right). \tag{7.6}$$

The equation of this type was considered in [1] from the point of view of the canonical quantization approach.

It is easy to prove that (7.6) obtained from the action principle is equivalent to the Heisenberg equations,

$$\dot{p}_\mu = \frac{1}{\mathrm{i}\hbar}[p_\mu, H], \quad H = \frac{1}{2}(p,p) = \frac{1}{2}(\dot{q},\dot{q}) = L. \tag{7.7}$$

The generator $G = p_\mu \circ \delta q^\mu$ conserves due to (7.6), i.e. $\dot{G} = 0$. This fact confirms that our quantization scheme is self-consistent.

8 Hamiltonian as generator of time shifts

Let us consider the coordinate transformation $q \to q + \delta q$ caused by the infinitesimal time shift $t \to t + \delta t(t)$:

$$\delta q^\mu = \dot{q}^\mu \delta t,$$
$$\delta \dot{q}^\mu = \frac{\mathrm{d}}{\mathrm{d}t}\delta q^\mu - \dot{q}^\mu \frac{\mathrm{d}}{\mathrm{d}t}\delta t. \tag{8.1}$$

Using the commutation relations for the dynamical variables $\{q^\mu\}$ and $\{\dot{q}^\mu\}$, we obtain the following operator properties of the variations (8.1):

$$[\dot{q}^\mu, \delta q^\nu] = [\dot{q}^\mu, \dot{q}^\nu]\delta t = [\delta q^\mu, \dot{q}^\nu]; \tag{8.2}$$
$$[q^\mu, \delta q^\nu] = [q^\mu, \dot{q}^\nu]\delta t = \mathrm{i}\hbar g^{\mu\nu}\delta t. \tag{8.3}$$

The variation of the Lagrangian L caused by these variations equals

$$\delta_t L = \delta L + L\delta t. \tag{8.4}$$

It is easy to rewrite (8.4) as a total time derivative without any referring to the equations of motion (i.e. in a purely algebraic way with the use of (8.1)):

$$\delta_t L = \frac{\mathrm{d}}{\mathrm{d}t}(L\delta t). \tag{8.5}$$

This fact leads to the variations (8.1) being permissible ones.

Further we transform (8.4) using explicitly the dynamical equations obtained above. By extracting a total time derivative we can write

$$\delta_t L = \frac{\mathrm{d}}{\mathrm{d}t}(L\delta t) + \frac{\mathrm{d}L}{\mathrm{d}t}\delta t - (\dot{p}_\mu - f_\mu) \circ \delta q^\mu. \tag{8.6}$$

The last term in (8.6) contains the dynamical equations (7.6). Comparing (8.5) with (8.6) we can observe that

$$\frac{\mathrm{d}H}{\mathrm{d}t}\delta t = 0, \quad H = L. \tag{8.7}$$

Therefore the Hamiltonian of our model is the conserved generator of time shifts.

9 Hilbert space of states

Let the coordinate operators $\{q^\mu\}$ form the complete set of commutative observables. We define its spectrum by the equation

$$\hat{q}^\mu |q\rangle = q^\mu |q\rangle \tag{9.1}$$

(to avoid confusion we write an operator with a hat and a c-number without it in this formula and in similar cases below).

The eigenvectors of $\{q^\mu\}$ are normalized by the following condition:

$$\langle q'' | q' \rangle = \frac{1}{\sqrt{g(q'')}} \delta(q'' - q') := \Delta(q'' - q'). \tag{9.2}$$

Here $\Delta(q)$ is the δ function on V_n in conformity with the volume element $\mathrm{d}V = (g(q))^{1/2} \mathrm{d}q$, $\mathrm{d}q = \mathrm{d}q^1 \ldots \mathrm{d}q^n$. This function has the following properties:

$$\int F(q')\Delta(q' - q)\mathrm{d}q = F(q), \tag{9.3}$$
$$\frac{\partial \Delta(q' - q)}{\partial q'^\mu} = -\Gamma_\mu(q')\Delta(q' - q) - \frac{\partial \Delta(q' - q)}{\partial q^\mu},$$

$$(F(q') - F(q))\Delta(q' - q) = 0, \tag{9.4}$$
$$(F(q') - F(q)) \frac{\partial \Delta(q' - q)}{\partial q'^\mu} = -\frac{\partial F(q')}{\partial q'^\mu} \Delta(q' - q),$$

for an arbitrary smooth function $F(q)$ on V_n.

To construct a coordinate representation associated with the complete set $\{q^\mu\}$, we need the matrix elements of the acting operators. For the coordinate operator q^μ we easily find

$$\langle q'' | q^\mu | q' \rangle = q'^\mu \langle q'' | q' \rangle . \tag{9.5}$$

In order to calculate the matrix element of the momentum operator let us consider the matrix element of the commutator $[q^\mu, p_\nu] = \mathrm{i}\hbar\delta^\mu_\nu$:

$$\langle q'' | q^\mu p_\nu - p_\nu q^\mu | q' \rangle = \mathrm{i}\hbar\delta^\mu_\nu \Delta(q'' - q'). \tag{9.6}$$

The left hand side of the equality (9.6) can be rewritten as

$$\begin{aligned}\langle q'' | [q^\mu, p_\nu] | q' \rangle &= \int \mathrm{d}V''' \left(\langle q'' | q^\mu | q''' \rangle \langle q''' | p_\nu | q' \rangle \right. \\ &- \left. \langle q'' | p_\nu | q''' \rangle \langle q''' | q^\mu | q' \rangle \right) \\ &= \int \mathrm{d}V''' \left(q'''^\mu \Delta(q'' - q''') \langle q''' | p_\nu | q' \rangle \right. \\ &- \left. q'^\mu \Delta(q''' - q') \langle q'' | p_\nu | q''' \rangle \right) \\ &= (q''^\mu - q'^\mu) \langle q'' | p_\nu | q' \rangle .\end{aligned}$$

Therefore, (9.6) is equivalent to

$$(q''^\mu - q'^\mu) \langle q'' | p_\nu | q' \rangle = \mathrm{i}\hbar\delta^\mu_\nu \Delta(q'' - q'). \tag{9.7}$$

Equation (9.7) can be viewed as the equation for the unknown $\langle q'' | p_\nu | q' \rangle$. Using the properties of the Δ function we obtain the solution of (9.7) in the form

$$\begin{aligned}\langle q'' | p_\mu | q' \rangle = &- \mathrm{i}\hbar \frac{\partial}{\partial q''^\mu} \Delta(q'' - q') \\ &+ F_\mu(q'')\Delta(q'' - q'),\end{aligned} \tag{9.8}$$

where $F_\mu(q)$ is some smooth function on V_n which will be determined later. Its appearance in (9.8) does not lead to any inner contradictions. To observe this, let us calculate the matrix element of the commutator $[f, p_\mu]$ for some operator f using (9.5), (9.8) and (9.3) and (9.4). We find

$$\begin{aligned}\langle q'' | [f, p_\mu] | q' \rangle &= (F_\mu(q') - F_\mu(q'')) \langle q'' | f | q' \rangle \\ &+ \mathrm{i}\hbar \left(\frac{\partial}{\partial q''^\mu} + \frac{\partial}{\partial q'^\mu} + \Gamma_\mu(q') \right) \langle q'' | f | q' \rangle .\end{aligned} \tag{9.9}$$

For the case $[q^\mu, f] = 0$ we have

$$\langle q'' | f(q) | q' \rangle = f(q')\Delta(q'' - q'). \tag{9.10}$$

Now (9.9) can be reduced to

$$\langle q'' | [f(q), p_\mu] | q' \rangle = \mathrm{i}\hbar \frac{\partial f(q')}{\partial q'^\mu} \Delta(q'' - q') \tag{9.11}$$

(using (9.3) and (9.4) and (9.10)). This result is completely in agreement with the commutator

$$[f(q), p_\mu] = \mathrm{i}\hbar \frac{\partial f(q)}{\partial q^\mu}.$$

Now we turn to the explicit form of the function $F_\mu(q)$. To determine it we need the matrix element of the commutator $[p_\mu, p_\nu] = 0$ which can be obtained by replacing $f(q)$ by p_ν in (9.9):

$$\langle q'' | [p_\mu, p_\nu] | q' \rangle = \mathrm{i}\hbar \left(\frac{\partial F_\nu(q')}{\partial q'^\mu} - \frac{\partial F_\mu(q')}{\partial q'^\nu} \right) \Delta(q'' - q') = 0.$$

From this equation we find

$$F_\mu = \frac{\partial F(q)}{\partial q^\mu}, \tag{9.12}$$

where $F(q)$ is some scalar function. Due to (9.12) we rewrite (9.8) as

$$\begin{aligned}\langle q'' | p_\mu | q' \rangle = &-\mathrm{i}\hbar \frac{\partial \Delta(q'' - q')}{\partial q''^\mu} \\ &+ \frac{\partial F(q'')}{\partial q''^\mu} \Delta(q'' - q').\end{aligned} \tag{9.13}$$

The hermitean conjugation of (9.13) due to (9.3) and (9.4) can be expressed as

$$\begin{aligned}\langle q' | p_\mu | q'' \rangle^* = &-\mathrm{i}\hbar \frac{\partial \Delta(q'' - q')}{\partial q''^\mu} - \mathrm{i}\hbar\Gamma_\mu(q'')\Delta(q'' - q') \\ &+ \frac{\partial F^*}{\partial q''^\mu} \Delta(q'' - q').\end{aligned} \tag{9.14}$$

The hermitean property of p_μ leads to

$$\langle q''|\, p_\mu \,|q'\rangle = \langle q'|\, p_\mu \,|q''\rangle^* \,;$$

then

$$\mathrm{Im}\left(\frac{\partial F}{\partial q^\mu}\right) = \frac{1}{2\mathrm{i}}\frac{\partial}{\partial q^\mu}(F - F^*) = \frac{\hbar}{2}\Gamma_\mu. \qquad (9.15)$$

From the definition of Γ_μ we find

$$\Gamma_\mu = \frac{1}{2}\partial_\mu \ln g.$$

Therefore we can decompose $F(q)$ into real and imaginary parts:

$$F = -\varphi - \frac{\mathrm{i}\hbar}{4}\ln g, \qquad (9.16)$$

where φ is some real-valued scalar function on V_n.

Using (9.16) in (9.8) we finally write the matrix element of p_μ:

$$\langle q''|\, p_\mu \,|q'\rangle = -\mathrm{i}\hbar\frac{\partial \Delta(q''-q')}{\partial q''^\mu} - \left(\frac{\partial\varphi}{\partial q''^\mu} + \frac{\mathrm{i}\hbar}{2}\Gamma_\mu(q'')\right)\Delta(q''-q'), \qquad (9.17)$$

depending on an arbitrary real-valued function $\varphi(q)$. Its appearance in (9.17) does not affect the physical states because we can eliminate $\varphi(q)$ by the unitary transformation

$$|q\rangle \to U(q)\,|q\rangle\,,$$
$$p_\mu \to U p_\mu U^\dagger = p_\mu - \partial_\mu\varphi, \qquad (9.18)$$
$$q^\mu \to U q^\mu U^\dagger = q^\mu, \qquad (9.19)$$

where

$$U(q) = \exp\left(-\frac{1}{\mathrm{i}\hbar}\varphi(q)\right)$$

(see [15]). Therefore, without loss of generality we assume $\varphi(q) = 0$.

Now we construct the coordinate representation for our model. In order to do this, we represent the wave function by

$$\psi(q) = \langle q\,|\psi\rangle \qquad (9.20)$$

for an arbitrary state $|\psi\rangle$.

The coordinate representation of the operator f is defined by the following formula:

$$\hat f\psi(q) := \langle q|\, f\,|\psi\rangle = \int \mathrm{d}q'\sqrt{g(q')}\,\langle q|\, f\,|q'\rangle\,\langle q'\,|\psi\rangle\,. \qquad (9.21)$$

Substituting $f = q^\mu$ we can obtain

$$\hat q^\mu\psi(q) = \int \mathrm{d}q'\sqrt{g(q')}\,\langle q|\, q^\mu\,|q'\rangle\,\langle q'\,|\psi\rangle = q'^\mu\psi(q). \qquad (9.22)$$

Similarly, making the substitution $f = p_\mu$ in (9.21) we write

$$\hat{p_\mu}\psi(q) = \int \mathrm{d}q'\sqrt{g(q')}\,\langle q|\, p_\mu\,|q'\rangle\,\langle q'\,|\psi\rangle = \int \mathrm{d}q'\sqrt{g(q')} \times \left(F_\mu(q')\Delta(q-q') - \mathrm{i}\hbar\frac{\partial\Delta(q-q')}{\partial q^\mu}\right)\psi(q) = -\mathrm{i}\hbar\frac{\partial\psi(q)}{\partial q^\mu} - \frac{\mathrm{i}\hbar}{2}\Gamma_\mu(q)\psi(q) - \frac{\partial\varphi(q)}{\partial q^\mu}\psi(q).$$

Taking $\varphi(q) = 0$, we finally have

$$\hat p_\mu\psi(q) = -\mathrm{i}\hbar\left(\frac{\partial}{\partial q^\mu} + \frac{1}{2}\Gamma_\mu(q)\right)\psi(q). \qquad (9.23)$$

This representation coincides with [16]. Under a general coordinate transformation the object (9.23) transforms as a vector.

Hence we have found the coordinate representation for the coordinate and momentum operators:

$$\hat q^\mu = q^\mu, \quad \hat p_\mu = -\mathrm{i}\hbar\left(\frac{\partial}{\partial q^\mu} + \frac{1}{2}\Gamma_\mu\right). \qquad (9.24)$$

The coordinate representation for the operators π_μ and $\pi_\mu^\dagger$ can be constructed in a similar way. Their matrix elements have the form

$$\langle q''|\, \pi_\mu\,|q'\rangle = -\mathrm{i}\hbar\frac{\partial}{\partial q''^\mu}\Delta(q''-q'), \qquad (9.25)$$
$$\langle q''|\, \pi_\mu^\dagger\,|q'\rangle = -\mathrm{i}\hbar\frac{\partial}{\partial q''^\mu}\Delta(q''-q') - \mathrm{i}\hbar\Gamma_\mu(q')\Delta(q''-q').$$

Therefore,

$$\hat\pi_\mu = -\mathrm{i}\hbar\frac{\partial}{\partial q^\mu}, \quad \hat\pi_\mu^\dagger = -\mathrm{i}\hbar\frac{\partial}{\partial q^\mu} - \mathrm{i}\hbar\Gamma_\mu. \qquad (9.26)$$

In order to find the coordinate representation of the Hamiltonian, we use the formula

$$\hat A(\hat B\psi) = \int \mathrm{d}V'\mathrm{d}V''\,\langle q|\, A\,|q'\rangle\,\langle q'|\, B\,|q''\rangle\,\langle q''\,|\psi\rangle = \int \mathrm{d}V''\,\langle q|\, AB\,|q''\rangle\,\langle q''\,|\psi\rangle = \widehat{AB}\psi. \qquad (9.27)$$

Putting $\hat A = \hat\pi_\mu^\dagger$ and $\hat B = g^{\mu\nu}\pi_\mu$ into (9.27), we have

$$2\hat H\psi = (\pi_\mu^\dagger g^{\mu\nu}\pi_\nu)^\wedge\psi = \hat\pi_\mu^\dagger\,(g^{\mu\nu}\hat\pi_\nu\psi) = -\hbar^2\left[\frac{\partial}{\partial q^\mu}\left(g^{\mu\nu}\frac{\partial\psi}{\partial q^\nu}\right) + \Gamma_\mu\left(g^{\mu\nu}\frac{\partial\psi}{\partial q^\nu}\right)\right] = -\hbar^2\frac{1}{\sqrt{g}}\frac{\partial}{\partial q^\mu}\left(g^{\mu\nu}\frac{\partial}{\partial q^\nu}\right)\psi = -\hbar^2\nabla_\mu g^{\mu\nu}\nabla_\nu\psi. \qquad (9.28)$$

Here ∇_μ is the covariant derivative in the metric $\{g_{\mu\nu}\}$.

Hence, the coordinate representation of the Hamiltonian

$$\hat{H} = -\frac{1}{2}\hbar^2 \nabla_\mu g^{\mu\nu} \nabla_\nu \psi \tag{9.29}$$

is nothing but the Laplace operator on V_n.

The Schrödinger equation for a free particle on V_n reads

$$-\frac{\hbar^2}{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q^\mu}\left(g^{\mu\nu}\frac{\partial\psi}{\partial q^\nu}\right) = E\psi. \tag{9.30}$$

Coming to the end of this section we add a remark on the form of the generator G. Using (9.27) we directly calculate

$$\hat{G} = \hat{v}^\mu \circ \hat{p}_\mu = -\mathrm{i}\hbar v^\mu \frac{\partial}{\partial q^\mu}, \tag{9.31}$$

where $\{v^\mu\}$ is a Killing vector (note that we need the particular form of the Killing equation, namely $\nabla_\mu v^\mu = 0$).

10 Conclusions

Observing the obtained results, we can conclude that our extension of Schwinger's quantization procedure allows one to solve the problem of the formulation of quantum mechanics on the manifold with a group structure without assuming non-strictly motivated assumptions.

The main features of the present work, which have a general character, are

(1) the logical motivation of the use of Killing vectors as permissible variations in quantum mechanics on the Riemannian space in Schwinger's approach;
(2) the method of construction of the Lagrangian which is invariant under a general coordinate transformation;
(3) the definition of the quantum norm of the velocity and momentum operators which is invariant under a general coordinate transformation.

Applying these, we have rigorously defined quantum mechanics on the manifold with a simply transitive group of isometries. The theory includes commutative relations, Lagrangian and Heisenberg equations of motion, and seems to be self-consistent.

These results, obtained within the framework of a unified quantization approach, are in accordance with [1,16], where the quantum theory was developed by means of canonical quantization methods based on some special assumptions.

In forthcoming papers we will apply our quantization procedure to construct a quantum theory on Riemannian manifolds with a more complicated group structure.

Appendix

A Symmetrized Jordan product

If a, b are hermitean operators, the product $a \cdot b$ is non-hermitean in the general case. The hermitean condition holds for the Jordan product

$$a \circ b = \frac{1}{2}(ab + ba). \tag{A.1}$$

From this definition one can immediately obtain

(1) $a \circ b = b \circ a$;
(2) $(a + b) \circ c = a \circ c + b \circ c$;
(3) $(\alpha a) \circ b = a \circ (\alpha b) = \alpha(a \circ b)$, $\alpha \in C$;
(4) $[a, b \circ c] = [a, b] \circ c + b \circ [a, c]$.

The Jordan product is non-associative:

$$a \circ (b \circ c) = (a \circ b) \circ c - \frac{1}{4}[b, [a, c]]. \tag{A.2}$$

Let us concentrate our attention on the combinations appearing in our model. The basic assumptions have the form (see Sect. 4)

(1) $\forall \mu, \nu = \overline{1, n}, \quad [q^\mu, q^\nu] = 0$,
(2) $\frac{1}{\mathrm{i}\hbar}[q^\mu, \dot{q}^\nu] = f^{\mu\nu}(q)$.

Taking the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}[q^\mu, q^\nu] = [\dot{q}^\mu, q^\nu] + [q^\mu, \dot{q}^\nu] + 0,$$

we find

$$f^{\mu\nu}(q) = f^{\nu\mu}(q). \tag{A.3}$$

Due to these properties, the time derivative of the operator $F(q)$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}F(q) = \dot{q}^\mu \circ \frac{\partial F(q)}{\partial q^\mu} \tag{A.4}$$

(if $F(q)$ is a polynomial, the proof is elementary).

It is important to note that in some cases, that are determined by the operator properties of the multipliers, the Jordan product is associative. Looking at the formula (A.2) we see that $a \circ (b \circ c) = (a \circ b) \circ c$ if $[a, b] = 0$ or $[b, [a, c]] = 0$. Taking into account the basic assumptions we can write

$$f_1(q) \circ (\dot{q}^\mu \circ f_2(q)) = (f_1(q) \circ \dot{q}^\mu) \circ f_2(q), \tag{A.5}$$

$$\begin{aligned} f_1(q) \circ (\dot{q}^\mu \circ f_2(q)) &= f_1(q) \circ (f_2(q) \circ \dot{q}^\mu) \\ &= (f_1(q) f_2(q)) \circ \dot{q}^\mu \\ &\quad - \frac{1}{4}[f_2(q), \text{some function of the } q\text{'s}] \\ &= (f_1(q) f_2(q)) \circ \dot{q}^\mu. \end{aligned} \tag{A.6}$$

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